APPLYING THE METHOD OF ADJOINT FIELDS TO NONLINEAR SYSTEMS

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An analogy whereby the study of the phase trajectories of a nonlinear self-contained system is replaced by the investigation of the trajectories of some field of forces called the "adjoint" field is considered in [1]. In the present paper we examine the properties of adjoint fields and their application to nonlinear self-contained systems.

1. Let us consider a two-dimensional self-contained system of the form

$$x^{\cdot} = p(x, y), \qquad y^{\cdot} = q(x, y) \tag{1.1}$$

where x and y are the coordinates of a point in the phase plane. We assume that the functions $p(x, y)$ and $q(x, y)$ are continuous and that they have continuous partial derivatives (with respect to both variables) of up to and including the second order in the given domain of variation of the variables x and y.

In seeking the derivatives $x^{\cdot\cdot}$ and $y^{\cdot\cdot}$, by virtue of (1.1) we obtain equations which can be considered as the equations of motion of a point of unit mass ($m = 1$) in the force field $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$

$$x^{\cdot\cdot} = \frac{\partial p}{\partial x} p + \frac{\partial p}{\partial y} q = P(x, y), \qquad y^{\cdot\cdot} = \frac{\partial q}{\partial x} p + \frac{\partial q}{\partial y} q = Q(x, y) \tag{1.2}$$

Following Liu and Fett [1], we shall call the resulting force field $\mathbf{F}$ the adjoint field of system (1.1) and Eqs. (1.2) the adjoint equations of motion.

Since $x^{\cdot} = p(x, y)$ and $y^{\cdot} = q(x, y)$ can be considered as partial integrals of system (1.2), the family of phase trajectories of self-contained system (1.1) is a subset of the set of all trajectories of system (1.2). Hence, the problem of finding the phase trajectories of system (1.1) can be reduced to the more general problem of finding the trajectories of adjoint system (1.2).

2. Any adjoint force field of the form (1.2) can always be normalized and represented as a superposition of two fields of which one is potential and the other a gyroscopic force field (this is the normalization theorem).

In order to convince ourselves, let us introduce the two functions

$$V(x, y) = -{}^1/_2\,(p^2 + q^2), \qquad \Omega(x, y) = \partial p / \partial y - \partial q / \partial x \tag{2.1}$$

Then, by (1.2), we obtain

$$P(x, y) = -\partial V / \partial x + \Omega q, \qquad Q(x, y) = -\partial V / \partial y - \Omega p \tag{2.2}$$

Hence it follows that a material particle in the adjoint field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is acted on by two forces: the conservative force formed by the field with the potential $V(x, y) = -{}^1/_2\,(p^2 + q^2) + \text{const}$ and by the gyroscopic force

$$\Gamma = \Gamma_x \mathbf{i} + \Gamma_y \mathbf{j} \qquad (\Gamma_x = \Omega y^{\cdot},\ \Gamma_y = -\Omega x^{\cdot}) \tag{2.3}$$

In fact, by (1.1) and (2.3), the work performed by the force Γ over any real displacement $ds(dx, dy)$ is equal to $\Gamma \cdot ds = \Omega\,(y^{\cdot}\,dx - x^{\cdot}\,dy) = 0$.

By virtue of (1.2) and (2.2), adjoint equations of motion (1.2) can be written in the form

$$x^{\cdot\cdot} - \Omega y^{\cdot} = -\,\partial V / \partial x, \qquad y^{\cdot\cdot} + \Omega x^{\cdot} = -\,\partial V / \partial y \tag{2.4}$$

Birkhoff [2] obtained the equations of motion in this form for a Lagrangian system with two degrees of freedom with a constant energy h equal to zero and a Lagrangian L given by the Expression $L = {}^1/_2\,(x^{\cdot 2} + y^{\cdot 2}) - p\,(x,\ y)\,x^{\cdot} - q\,(x,\ y)\,y^{\cdot} - V\,(x,\ y)$

where $V(x, y)$ is defined by (2.1).

3. Let us find the condition which the given functions $p\,(x, y)$ and $q\,(x, y)$ must satisfy in order for the adjoint force field (1.2) to be conservative. Clearly, it is conservative if and only if $\partial P / \partial y = \partial Q / \partial x$.

Hence, after some simplifications we find from (1.2) and (2.1) that

$$\partial\,(\Omega p) / \partial x + \partial\,(\Omega q) / \partial y = 0 \tag{3.1}$$

Thus, condition (3.1) guaranteeing the conservativeness of the adjoint force field is equivalent to the condition of continuous steady flow of an ideal fluid, provided the velocities of the phase point $x^{\cdot} = p\,(x, y)$ and $y^{\cdot} = q\,(x, y)$ are considered as the velocities of a fluid particle ; the role of the fluid density ρ is played by the function Ω, which, as we see, is the negative vortex strength $\zeta = q_x - p_y$.

Let us consider some particular cases when condition (3.1) is fulfilled.

A) The case (*) $\Omega = 0$ (where the condition of the total differential of the Pfaffian form $p dx + q dy$ is fulfilled) has a simple hydrodynamic interpretation, i.e. that the motion of the fluid associated with the motion of system (1.1) is nonvortical ($\zeta = -\Omega = 0$).

B) The case $\Omega\,(x,\ y) = C\ \ (C \neq 0)$ also has a simple hydrodynamic interpretation, i.e. the corresponding fluid flow is vortical with a constant vortex velocity $\zeta(x, y) = q_x - p_y = \text{const}$.

By virtue of condition (3.1) we obtain

$$\partial p / \partial x + \partial q / \partial y = 0, \ \partial p / \partial y - \partial q / \partial x = C \tag{3.2}$$

Relations (3.2) can be written in the form of Cauchy-Riemann conditions. This enables us to formulate the following statement.

In order for the adjoint field of self-contained system (1.1) under the condition $\Omega = C$ to be conservative it is necessary and sufficient that one of the functions

$$F_1\,(z) = p\,(x,\ y) - i\,(q\,(x, y) + Cx) \quad \text{or} \quad F_2\,(z) = (p\,(x, y) - Cy) - iq\,(x,\ y)$$

be analytic. Here the functions $p(x, y)$ and $q\,(x, y)$ are harmonic but not conjugate.

We introduce the function $H(x, y)$, setting

$$\partial H / \partial y = p\,(x,\ y), \qquad \partial H / \partial x = -\,q\,(x,\ y) \tag{3.3}$$

By (3.2), the function $H(x, y)$ must satisfy the Poisson Eq. $\Delta H(x, y) = C$. It plays the role of the Hamiltonian for initial system (1.1),

$$dx / dt = \partial H / \partial y, \qquad dy / dt = -\,\partial H / \partial x$$

Along the phase trajectories we have $q dx - p dy = 0$. Hence, by virtue of (3.3), we obtain $dH = 0$, so that $H(x, y) = \text{const}$ is the integral of system (1.1). Finally, for the

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*) Case A is considered in [1].

adjoint field we obtain

$$P = -\partial V^* / \partial x, \quad Q = -\partial V^* / \partial y \quad (V^* = -{}^1/_2 (p^2 + q^2) + CH) \qquad (3.4)$$

C) The case where $\Omega(x, y)$ is variable. Condition (3.1) of conservativeness of the adjoint field expresses the fact that $\Omega(x, y)$ is the integrating factor of the differential equation of the phase trajectories of system (1.1). Condition (3.1) can be satisfied by setting

$$\frac{\partial Y}{\partial x} = -\Omega q, \qquad \frac{\partial Y}{\partial y} = \Omega p \qquad (3.5)$$

By (3.5), along the phase trajectories we have $dY = \Omega(-pq + qp)\, dt = 0$, so that $Y(x, y) = \text{const}$ is the integral system (1.1). Here the potential of the adjoint field is $V^*(x, y) = -{}^1/_2 (p^2 + q^2) + Y(x, y)$.

4. A self-contained system of the form

$$x^\cdot = q(x, y), \qquad y^\cdot = -p(x, y) \qquad (4.1)$$

will be called conjugate to the basic system (1.1).

The phase trajectories of basic system (1.1) and conjugate system (4.1) are clearly orthogonal. Conjugate system (4.1) is likewise subject to a theorem on the normalization of the adjoint force field analogous to the theorem proved for basic system (1.1).

The condition of conservativeness of the adjoint force field for conjugate system (4.1) is of the form

$$\partial (q\Omega^*) / \partial x - \partial (p\Omega^*) / \partial y = 0 \qquad (\Omega^* = \partial p / \partial x + \partial q / \partial y) \qquad (4.2)$$

This expression is at the same time the condition under which the function $\Omega^*(x, y)$ is the integrating factor of the differential equation of the trajectories of system (4.1),

$$p(x, y)\, dx + q(x, y)\, dy = 0$$

Here we can also consider the cases $\Omega^* = 0$, $\Omega^* = \text{const}$, giving them a simple hydrodynamic interpretation as we did in the case of system (1.1).

If we require simultaneous fulfilment of conditions (3.1) and (4.2) of the conservativeness of the basic and conjugate systems, it becomes necessary to consider simultaneously Eqs.

$$\frac{\partial \Omega}{\partial x} p + \frac{\partial \Omega}{\partial y} q = -\Omega (p_x + q_y), \qquad \frac{\partial \Omega^*}{\partial x} q - \frac{\partial \Omega^*}{\partial y} p = -\Omega^* (q_x - p_y)$$

which by virtue of (1.1), (4.1), as well as (2.1) and (4.2), can be written in the form

$$d\Omega / dt = -\Omega\Omega^*, \qquad d\Omega^* / dt = \Omega\Omega^* \qquad (4.3)$$

This implies that $\Omega + \Omega^* = \text{const}$.

5. Let us turn again to self-contained system (1.1). Let the functions $p(x, y)$ and $q(x, y)$ be such that condition (3.1) of the conservativeness of the adjoint force field is not fulfilled, and let $\Omega(x, y) = p_y - q_x \neq 0$.

We pose the problem of finding a transform of self-contained system (1.1), and hence of the adjoint force field, which would render the adjoint field conservative without altering the phase trajectory picture. We shall tackle the problem by means of the reducing factor method proposed by Chplygin [3] in his consideration of nonholonomic systems.

We introduce the new independent variable τ by way of the relation

$$dt = \omega(x, y)\, d\tau \qquad (5.1)$$

Here $\omega(x, y)$ is some appropriately chosen function of the variables x and y which is called the reducing factor. A transform of the (5.1) type was also used by Birkhoff in his study of Lagrangian systems.

With the aid of the indicated transform (5.1), system (1.1) can be written as

$$dx / d\tau = p(x, y)\,\omega(x, y), \qquad dy / d\tau = q(x, y)\,\omega(x, y) \tag{5.2}$$

while the differential equation of the phase trajectories retains its original form $p(x, y)\,dy - q(x, y)\,dx = 0$. Expressing for brevity the right sides of (5.2) in terms of $p^*(x, y)$ and $q^*(x, y)$, respectively, we write out the adjoint equations of motion as we did in the case of basic system (1.1). As a result we obtain

$$x^{\cdot\cdot} = -\partial V^* / \partial x + q^*\Omega^*, \qquad y^{\cdot\cdot} = -\partial V^* / \partial y - p^*\Omega^* \tag{5.3}$$

$$V^* = -{}^1/_2\omega^2 (p^2 + q^2), \qquad \Omega^* = \omega\Omega + (-q\,\partial\omega / \partial x + p\partial\omega / \partial y) \tag{5.4}$$

We choose the reducing factor $\omega(x, y)$ in such a way that $\Omega^* = p_y^* - q_y^* = 0$

The adjoint force field then becomes conservative, and for finding $\omega(x, y)$ we have from (5.4) the following partial differential Eq.:

$$-q\,\partial\omega / \partial x + p\,\partial\omega / \partial y + \omega\Omega = 0 \tag{5.5}$$

which, as we know, can lead to integration of the system

$$\frac{dx}{-q} = \frac{dy}{p} = \frac{d\ln\omega}{-\Omega} \tag{5.6}$$

It is easy to see that the reducing factor $\omega(x, y)$ is in this case the integrating factor of the differential equation of the phase trajectories of conjugate system (4.1).

6. Let us consider a generalized Van der Pol equation of the form

$$x^{\cdot\cdot} + \mu F(x)x^{\cdot} + k^2 x = 0 \qquad (\mu = \text{const}) \tag{6.1}$$

Here $F(x)$ is a given function of the variable x. We introduce the new variable $y = x^{\cdot} + G(x)$, where $G(x)$ is an appropriately chosen function of the variable x. Differentiating y and choosing $G(x)$ in such a way as to guarantee fulfilment of the condition $G'(x) = \mu F(x)$, by virtue of Eq. (6.1) we arrive at a self-contained system of the form

$$x^{\cdot} = y - G(x), \qquad y^{\cdot} = -k^2 x \tag{6.2}$$

Hence, the problem can be reduced to the study on the Liénard plane Eq.

$$\frac{dx}{dy} = \frac{G(x) - y}{k^2 x} \qquad \left(G(x) = \mu \int F(x)\,dx\right)$$

Let us apply the method of adjoint field to the study of system (6.2). We have

$$p = y - G(x), \qquad q = -k^2 x, \qquad \Omega = p_y - q_x = 1 + k^2$$

Let us consider the possible cases.

1) The adjoint force field is conservative, so that condition (3.1) is fulfilled. Since in this case $\Omega = \text{const}$, the functions $p(x, y)$ and $q(x, y)$ must satisfy the Laplace Eq. $\Delta p(x, y) = \Delta q(x, y) = 0$.

This yields $G''(x) = 0$, so that $F(x) = \text{const}$, i.e. Eq. (6.1) describes a linearly damped oscillator.

2) The adjoint force field is not conservative, so that

$$\partial(\Omega p) / \partial x + \partial(\Omega q) / \partial y \neq 0$$

Let us make use of the reducing factor method and write out system (5.6), which in this case is of the form

$$\frac{dx}{k^2 x} = \frac{dy}{y - G(y)} = \frac{d\ln\omega}{-(1 + k^2)} \tag{6.3}$$

Taking the extreme terms, integrating, and clearing logarithms, we obtain the following expression for the reducing factor $\omega(x)$:

$$\omega(x) = cx^{-\alpha} \qquad (\alpha = (1 + k^2) / k^2, \ c = \text{const})$$

The reducing factor $\omega(x)$ is the integrating factor of Eq. $(y - G(x))\, dx - k^2 x dy = 0$, which is the differential equation of the phase trajectories for the system conjugate to system (6.2). Determining y from (6.3) becomes simply a matter of integrating a first-order linear equation, and as a result we find

$$y = x^{1/k^2}\left(C_1 - \int \frac{G(x)}{k^2} x^{-\alpha} dx\right) \qquad \left(\alpha = \frac{1+k^2}{k^2}\right) \tag{6.4}$$

Here C_1 is an integrating constant. Thus, the system of lines orthogonal to the resulting trajectories (6.4) serves as the system of phase trajectories for the initial self-contained system (6.2).

As a further illustration let us consider the linear Eq.

$$x^{\cdot\cdot} + (A + Bx)G(x^{\cdot}) + \alpha x = 0 \tag{6.5}$$

to which many problems in self-oscillation theory are reducible.

Here A, B and α are constants, and $G(x^{\cdot})$ is some continuous function of the variable $x^{\cdot}$. The phase trajectory picture (6.5) can be described by means of the system

$$x^{\cdot} = y, \quad y^{\cdot} = -\alpha x - (A + Bx)G(y) \tag{6.6}$$

Let the adjoint force field be nonconservative, so that condition (3.1) is not fulfilled. We apply the reducing factor method. Since

$$p = y, \qquad q = -\alpha x - (A + Bx)\, G(y), \qquad \Omega(x, y) = 1 + \alpha + BG(y)$$

by virtue of (5.6) we obtain

$$\frac{dx}{\alpha x + (A + Bx)\, G(y)} = \frac{dy}{y} = \frac{d \ln \omega}{-(1+\alpha) - BG(y)}$$

Taking the mean and extreme terms, integrating, and clearing logarithms, we obtain the following expression for the reducing factor $\omega(y)$:

$$\omega(y) = y^{-(1+\alpha)} F(y) \qquad \left(F(y) = \exp\left(-B \int \frac{G(y)}{y} dy\right)\right) \tag{6.7}$$

Since the reducing factor $\omega(y)$ is also the integrating factor of the differential equation of the trajectories of the system conjugate to system (6.6), the Pfäffian form

$$y\, \omega(y) dx - \omega(y)\, (\alpha x + (A + Bx)\, G(y)) dy = 0$$

is the total differential of some function $\psi(x, y)$. Integrating, we find (by (6.7)) that

$$\psi(x, y) = y^{-\alpha} x F(y) - h(y) = \text{const} \tag{6.8}$$

$$h(y) = A \int y^{-(1+\alpha)} G(y)\, F(y)\, dy \tag{6.9}$$

Having constructed the system of lines orthogonal to the system $\psi(x, y) = \text{const}$ (6.8), we obtain the phase representation for the initial self-contained system (6.6).

In conclusion we consider a special case of Eq. (6.5),

$$x^{\cdot\cdot} + G(x^{\cdot}) + \alpha x = 0 \tag{6.10}$$

The phase trajectories can be obtained by considering a self-contained system of the form

$$x^{\cdot} = y, \qquad y^{\cdot} = -\alpha x - G(y) \tag{6.11}$$

Making use of (6. 7) and thus setting $A = 1$ and $B = 0$, we obtain

$$F(y) = 1, \quad \omega(y) = y^{-(1+\alpha)}$$

Thus, by virtue of (6. 8) and (6. 9), we obtain the expression for the phase trajectories of the system conjugate to (6. 11)

$$\psi(x, y) = xy^{-\alpha} - h(y) = \text{const} \qquad \left(h(y) = \int y^{-(1+\alpha)} G(y)\, dy\right)$$

Having constructed the system of lines orthogonal to system $\psi(x, y) = \text{const}$, we obtain the phase trajectory picture for initial system (6. 11).

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